Written Exam Economics Winter 2019-20

Auctions: Theory and Practice

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This exam question consists of 7 pages in total (including this front page)

Answers only in English.

The paper must be uploaded as <u>one PDF document</u>. The PDF document must be named with exam number only (e.g. '1234.pdf') and uploaded to Digital Exam.

Introduction

Throughout the assignment, please show your work. Simply stating the correct answer without sufficiently explaining your calculations/reasoning is not enough to get full credit. Correspondingly, an incorrect or incomplete answer that uses some of the correct argumentation may be given partial credit.

If you believe that there may be a typo in one of the questions, or if something is stated unclearly, please let us know as quickly as possible by sending an email to both Neil and Holger. Any material responses to such queries will be published on Absalon.

Good luck!

Problem 1 (True or false)

Please state whether each of the following statements is true or false and show the arguments and/or calculations which justify your conclusion.

1a. 5 bidders with private values uniformly distributed between 0 and 1 partake in a 1st price auction. Assuming that everyone is playing the symmetric equilibrium bidding strategy, the optimal bid for a bidder who makes a draw of 0.75 is 0.7.

- False.
- We know that the symmetric equilibrium bidding strategy in a private value first-price auction is to bid $E(Y_1 | \, Y_1 < x)$.
- We know that with 5 bidders uniformly distributed between 0 and 1, $E(Y_1 | \, Y_1 < x)$ is equivalent to $\underline{x} + \frac{(N-1)+1-k}{(N-1)+1}(x - \underline{x}) = 0 + \frac{(5-1)+1-1}{(5-1)+1}(x - 0) = \frac{4}{5}x$
- This means that a bidder which draws 0.75 should bid $\frac{4}{5} * 0.75 = 0.6$, which is not 0.7.

1b. The expected revenue of a 1st price auction with 2 bidders uniformly distributed between 2 and 6 and a reserve price of 3 is 3.5.

- False.
- We can calculate expected revenue for this first-price auction by calculating expected revenue for an equivalent second-price auction – because we know that the revenue equivalence theorem applies (so the calculation for a second-price auction must yield the same result as had we calculated for a first-price auction). We do this because it is easier to calculate expected revenue for a second-price auction.
- We calculate expected revenue in a corresponding second-price auction with a reserve price r and 2 bidders by summing revenue across 3 cases, weighted by the probability of each case occurring:
    1. Both bidders draw below r – revenue is 0
    2. Exactly one bidder draws above r – revenue is r
    3. Both bidders draw above r – expected revenue is equivalent to the expected second highest value (which mechanically determines the price in a second-price auction)
- Case 1 contributes to total expected revenue with exactly 0.

- Case 2 occurs with probability $F(r) * (1 - F(r)) * 2$ since exactly one of the two bidders must draw below the reserve price, exactly one must draw above, and this can occur in two ways (either the first bidder draws above and the second one draws below, or vice-versa). This probability can be calculated as:
$$F(r) * (1 - F(r)) * 2 = F(3) * (1 - F(3)) * 2 = \left(\frac{3}{6-2}\right) * \left(1 - \frac{3}{6-2}\right) * 2 = \frac{1}{4} * \frac{3}{4} * 2 = \frac{6}{16} = \frac{3}{8}$$
Revenue in this case is equal to the reserve price, which is 3. This means that the contribution of case 2 to total expected revenue, weighted by its probability, is $\frac{3}{8} * 3 = \frac{9}{8}$.
- Case 3 occurs with probability $(1 - F(r))^2$, since both of the two bidders must draw above the reserve price. This probability can be calculated as:
$$(1 - F(r))^2 = (1 - F(3))^2 = \left(1 - \frac{3}{6-2}\right)^2 = \left(\frac{3}{4}\right)^2 = \frac{9}{16}$$
Revenue in this case is the expected second highest draw. We know that the expectation of the kth-highest draw among N bidders that are uniformly distributed between $\underline{x}$ and $\overline{x}$ is $\underline{x} + \frac{N+1-k}{N+1}(\overline{x} - \underline{x})$. In this case, we have 2 bidders distributed between r and $\overline{x}$ (since we are considering the case where both have drawn above r), and we are interested in the expectation of the second-highest draw:
$$\underline{x} + \frac{N+1-k}{N+1}(\overline{x} - \underline{x}) = 3 + \frac{2+1-2}{2+1}(6-3) = 3 + \frac{1}{3} * 3 = 4$$
This means that the contribution of case 3 to total expected revenue, weighted by probability, is $\frac{9}{16} * 4 = \frac{9*4}{4*4} = \frac{9}{4}$.
- Total expected revenue is the sum of the three cases: $0 + \frac{9}{8} + \frac{9}{4} = \frac{9}{8} + \frac{18}{8} = \frac{27}{8} = 3.375$, which is not 3.5.

1c. In a multi-unit setting, a discriminatory auction is less susceptible to collusion than a Vickrey auction. (a qualitative explanation is sufficient)

- True.
- A discriminatory auction is a multi-unit auction format in which the winning bidders must pay prices as specified in their own bids. This is a kind of first-price mechanism which means that bidders have an incentive to shade their bids relative to their values in order to generate surplus.
- A Vickrey auction is a multi-unit auction format in which the winning bidders must pay prices as determined by the losing bids. Essentially, for each unit that they win, winners must pay a price which is only just high enough outbid the losing bidders. This is a kind of second-price mechanism which means that bidders have an incentive to simply bid their values (at least in a private value setting).
- Any auction with a first-price mechanism is likely to be less susceptible to collusion since bidders participating in a bidding ring will have an incentive to deviate from the bidding strategy that they would coordinate prior to the auction. This makes a bidding ring less likely to arise in the first place (since the ring members will not be able to trust one another). The incentive to deviate arises from the fact that the ring leader(s) will need to shade their bids in order to generate profit, which means that other members of the ring may be able to profitably outbid the ring leader(s).
- Any auction with a second-price mechanism is comparably more susceptible to collusion since bidders participating in a bidding ring will have no such incentive to deviate from the bidding strategy that they would coordinate prior to the auction. The ring leader(s), who have the highest values, will submit bids equivalent to their values and other members of the ring would thus have nothing to gain from deviating from the coordinated bidding strategy.

1d. An information rule can be used to avoid or mitigate so-called "push bidding". (a qualitative explanation is sufficient)

- False.
- "Push bidding" is a type of strategic bidding which is sometimes pursued by bidders in second-price auctions. The usual second-price (private value) logic dictates that bidders should submit bids equivalent to their values since this maximises their own payoff. If, however, bidders are also concerned with minimising the payoffs of competing bidders, then they might submit bids above their own valuations in order to increase the price of their competitors (without affecting their own payoff). "Push bidding" comes with the risk of overpaying if a "push bid" accidentally wins.
- An information rule is a type of rule which is employed in multi-round auctions to specify the amount of information that is shared with bidders during the auction (usually implemented via auction software). The auctioneer may, for example, decide to limit the amount of information that is provided regarding the identities of other bidders participating and bidding in the auction, or regarding the exact extent of aggregate demand. This can help to deter signalling and demand reduction.
- An information rule is not an effective tool against "push bidding". This is because the incentive to engage in "push bidding" depends almost exclusively on the existence of a second-price rule and not on the kind of information that could be provided during a multi-round auction. For example, it is apparent that "push bidding" can occur in a sealed-bid context where an information rule is not even relevant.
- *There may be some unusual circumstances in which an information rule could plausibly deter "push bidding", if, for example, a bidder is only interested in increasing the price of a specific competitor and is unable to verify/check whether they are still bidding in a multi-round auction due to a restrictive information rule. We have given credit if the student has answered "true" and clearly explained a correct set of circumstances which can justify this response.*

1e. In a multi-unit uniform auction, it is a dominant strategy for bidders with single-unit demand to bid their true value vector. (a qualitative explanation is sufficient)

- True.
- With private values and single-unit demand, it is indeed a (weakly) dominant strategy for bidders with single-unit demand to bid their true value vector, i.e. to bid their value for the single unit that they demand and then a vector of zeroes for all other units for sale.
- This is because we know that the optimal bidding strategy in a multi-unit uniform auction is to bid your value on the first unit and to shade your bids (i.e. to bid closer to 0) on all subsequent units. This balances the trade-off between winning additional units and minimising the price if your bids might become price-setting.
- In the case where the true values of units beyond the first unit are 0, then the bidder has no interest in winning additional units anyway, and hence the "optimal shade" is simply to bid 0, which is equivalent to the true values.
- *We have also accepted an answer of "false" if the student has explicitly referred to a case of interdependent values – in which case bidders would also have an incentive to shade their bid even on the first unit to avoid the winner's curse.*

Problem 2

Renowned graffiti artist Banksta has created a new painting. It depicts two monkeys in suits eating caviar. The painting serves as a subtle criticism of excessive wealth in the West.

Banksta wants to allocate the painting via auction to ensure efficiency (i.e. to make sure that the person who values the painting the most wins it). But he wants to avoid a high selling price as this would go against the very ideology of his painting.

Banksta thus devises an auction format whereby the highest bidder will win but pay a price only equivalent to the lowest bid submitted at the auction. Banksta believes that this format will guarantee efficiency whilst also achieving a low selling price.

We can think of this as an N-th price auction, i.e. the highest bidder wins but pays a price equivalent to the N-th highest bid (where N bidders partake in the auction).

Consider N risk-neutral Banksta fans with valuations x independently and uniformly distributed between 0 and 5 million pounds.

2a. Find the symmetric equilibrium bidding strategy for the N bidders in a sealed-bid N-th price auction. Please also explain the intuition of your result.

- Assuming the existence of a symmetric equilibrium bidding strategy $\beta^N(x)$, we know that we can formulate the expected payment in a sealed-bid N-th price auction as: $m_i^N(x) = prob(i\ wins|x) * E(\beta^N(Y_{N-1})|i\ wins, x)$, i.e. the payment is determined by the equilibrium bid of the bidder who draws Y_{N-1} (the lowest draw among the remaining $N-1$ bidders, conditional on winning).
- We know that the revenue equivalence theorem equates expected revenue in an N-th price auction with expected revenue in e.g. a second-price auction. So, expected payments for a bidder with the same valuation must be the same in both auction types.
- We know that bidders in a second-price auction have an expected payment which is: $m_i^{II}(x) = prob(i\ wins|x) * E(payment|i\ wins, x) = prob(i\ wins|x) * E(\beta^{II}(Y_1)|i\ wins, x)$
- The revenue equivalence theorem means that we can equate the two payments, i.e.: $m_i^N(x_i) = m_i^{II}(x_i) = prob(i\ wins|x) * E(\beta^N(Y_{N-1})|i\ wins, x) = prob(i\ wins|x) * E(\beta^{II}(Y_1)|i\ wins, x)$ $\Leftrightarrow E(\beta^N(Y_{N-1})|Y_1 < x) = E(\beta^{II}(Y_1)|Y_1 < x)$
- We know that $\beta^{II}(Y_1) = Y_1$ and we know that the expectation of Y_1, conditional on winning (for a uniform distribution between 0 and 5 million with N bidders), is: $E(Y_1) = \underline{x} + \frac{(N-1)+1-1}{(N-1)+1}(x_i - \underline{x}) = 0 + \frac{N-1}{N}(x_i - 0) = \frac{N-1}{N}x_i$
- We know that the expectation of Y_{N-1}, i.e. the lowest draw among other bidders, conditional on winning (for a uniform distribution between 0 and 5 million with N bidders), is: $E(Y_{N-1}) = \underline{x} + \frac{(N-1)+1-(N-1)}{(N-1)+1}(x_i - \underline{x}) = 0 + \frac{1}{N}(x_i - 0) = \frac{1}{N}x_i$
- Inserting these results into our equation gives us:
$$\beta^N\left(\frac{1}{N}x_i\right) = \frac{N-1}{N}x_i$$
- Under the assumption that $\beta^N(x_i)$ takes the form $\alpha^N * x_i$, we can now back out the value of α^N:
- $\alpha^N * \frac{1}{N}x_i = \frac{N-1}{N}x_i \Leftrightarrow \alpha^N * \frac{1}{N} = \frac{N-1}{N} \Leftrightarrow \alpha^N = N-1$
- This gives us the result that $\beta^N(x_i) = (N-1)x_i$
- Intuition: In equilibrium, bidders submit bids which are above their own value for all $N > 2$. For example, inserting for $N = 3$ yields the result that a bidder should bid double his/her value. This is

similar to the result found in class for third-price auctions, although the extent to which bidders should bid above their own value is even greater in this case.

- The reason why it is optimal for bidders to bid above their own value is that this strategy increases the probability of winning whilst being associated with an expected payment which is still below their own value, since a bidder can only win in the case where it has drawn the highest value, in which case the expected draw of the lowest-value bidder is very low (although the bid submitted by the lowest-value bidder will of course also be higher than its own value).
- Bidding above your own value does come with a risk of over-paying, although for bids up to $(N-1)x_i$, it must be that the expected benefits associated with an increased probability of winning at a price below your value outweigh the risk of over-paying (at least for a risk-neutral bidder).
- It is worth noting that the N-th price bidding strategy corresponds to the usual result of bidding your value in a second-price auction for $N = 2$, and also prescribes bidding 0 in the case where $N = 1$. Both of these results are also intuitive.

Banksta expects that the more people he can get to turn up at the auction, the lower the selling price will be, since it is more likely that low-value bidders will turn up and thus submit the lowest bid.

2b. Is Banksta correct that more bidders will lead to lower revenue? In other words, what is expected revenue in this auction, and how does it depend on the number of bidders? Please also explain the intuition of your result.

- Banksta is not correct that more bidders will lead to lower revenue.
- We can use the revenue equivalence theorem to calculate expected revenue in this auction and investigate how it depends on N.
- We know that expected revenue in a second-price auction is equal to: $E(R) = Y_2^N$, i.e. it is equal to the expected second-highest draw.
- Due to the revenue equivalence theorem, we know that expected revenue must be the same in the N-th price auction (under the assumption that all bidders play the symmetric equilibrium bidding strategy).
- For N bidders uniformly distributed between 0 and 5 million pounds, we find that $E(R)$ is equal to $Y_2^N = \underline{x} + \frac{N+1-2}{N+1}(\bar{x} - \underline{x}) = 0 + \frac{N-1}{N+1}(5-0) = \frac{N-1}{N+1}5$ million pounds.
- Deriving this with respect to the number of bidders N, we find $\frac{\partial E(R)}{\partial N} = \frac{10}{(N+1)^2}$, which is positive, i.e. we find that revenue depends positively, not negatively, on the number of bidders.
- Intuitively, Banksta must be right that it is more likely that low-value bidders will turn up at the auction as N increases.
- This effect does, *ceteris paribus*, imply a lower winning price.
- However, the equilibrium bidding strategy depends positively on the number of bidders (since $N-1$ depends positively on N), and it is clear from the aggregate result that this effect must outweigh the effect that Banksta had in mind.
- At the end of the day, more competition leads to higher revenues regardless of auction format.

Banksta concedes that he does not know exactly how many bidders will participate at the auction, and he realises that bidders will probably also be unsure regarding the true number of bidders when submitting their bid. He does not think that this should matter too much for achieving his main goal of efficiency.

2c. Do you think that the N-th price auction is likely to achieve an efficient outcome? Given Banksta's main goal of efficiency, would you recommend any changes to the auction design? (a qualitative explanation is sufficient)

- The N-th price auction is very unlikely to produce an efficient outcome (where the highest value bidder is the one that wins) if bidders are uncertain about the number of bidders participating in the auction.
- Each bidder's optimal bidding strategy depends on the number of bidders that it expects to face. In the symmetric equilibrium bidding strategy, the extent to which you should bid above your value increases depending on the number of other bidders at the auction.
- If all bidders were to play the symmetric equilibrium bidding strategy and if all bidders knew the number of bidders at the auction, then the auction would certainly produce an efficient outcome.
- The auction would still produce an efficient outcome, even with uncertainty regarding the number of bidders, if all bidders held the same beliefs regarding the different potential different numbers of bidders and the probabilities of each number occurring.
- It is, however, extremely unlikely that all bidders would hold the same beliefs regarding the different potential different numbers of bidders and the probabilities of each number occurring, especially if particular bidders are more certain to participate and others are more of a surprise (since each bidder would know whether they themselves will participate).
- If different bidders hold different beliefs about the likely number of bidders, then an inefficient outcome could very well arise, e.g. a situation where the bidder with the highest value underestimated the true number of bidders and hence bid "too low" whilst a bidder with a lower value overestimated the true number of bidders and hence bid "too high".

Problem 3

Horse-house is a famous auction house which sells racehorses. Horse-house has used the single-unit Dutch auction format for generations. Horse-house's main objective is revenue maximisation.

A risk-averse accountant has informed Horse-house's management that it should try to steady its revenue streams in coming years. To this aim, the accountant has suggested that Horse-house should consider switching to English auctions instead of Dutch auctions as this may lead to less variable revenue.

Horse-house's management asks you, the stable boy/girl (who has studied auction theory), what you think of the accountant's advice.

3a. Is the accountant correct that English auctions will generate more stable (i.e. less variable) revenue than Dutch auctions? Why/why not? (a qualitative explanation is sufficient)

- The accountant is not correct that English auctions will generate more stable revenue streams.
- As a point of departure, in a private value setting, there is revenue equivalence between Dutch auctions (where the price starts high and descends until a single bidder submits a bid) and English auctions (where the price starts low and increases until just a single bidder is left). This means that both formats will yield the same revenue on average.
- However, even though the two auction formats will yield the same revenue on average, realised revenue can and will differ between the two formats for specific bidders and draws – and the distribution of realised revenue around the average can thus also differ between the two formats.
- In practice, we know that Dutch auctions will yield more stable revenue than English auctions, i.e. Dutch auctions will have a tighter distribution of realised revenue around the same average.
- This is because we know that:
    a) Dutch auctions hold strategic equivalence with sealed-bid first-price auctions, whilst English auctions hold strategic equivalence with sealed-bid second-price auctions (when bidders hold private values).

b) First-price auctions have a tighter distribution of realised revenues because bidders shade, meaning that revenue is distributed from $\underline{x}$ to $E(Y_1|x = \omega)$ whereas in second-price auctions the distribution of revenue spans from $\underline{x}$ to ω (which is higher than $E(Y_1|x = \omega)$).

- If the accountant wants more stable revenue streams, it is thus wise to stick with the Dutch format.

The accountant also advises management that there is a common value element to the valuations of racehorses (since the value of a horse depends in part on its forecasted ability to win races). The accountant advises that English auctions may lead to higher revenues in such a situation.

Again, Horse-house's management asks you, the stable boy/girl, for your thoughts.

3b. Given the common value element (with affiliated signals) of racehorse valuations, would you expect English auctions to generate more, less or the same revenue as Dutch auctions? Why? (a qualitative explanation is sufficient)

- The accountant is correct that English auctions will generate more revenue if there is a common value element to racehorse valuations.
- As previously discussed, the Dutch auction and the English auction will yield the same revenue on average in a private value setting.
- This revenue equivalence breaks down, however, in a common value setting (with affiliated signals).
- This is because bidders can condition their bid strategy on the information that they receive during an English auction. Because other bidders' signals are affiliated with their own, they will be able to update their expectation of the value of the item conditional on winning as the auction progresses, , and consequently remove the winner's curse "risk premium" from their bids.
- Bidders are not able to update their bid strategy in the same way in a Dutch auction, since bidders cannot observe any useful information regarding other bidders' signals as the auction progresses (all bidders simply bid for 0 until the auction suddenly ends). This means that bidders cannot observe any signals which could help them remove the "risk premium" from their bids.
- The ability to remove the "risk premium" means that bidders will ultimately submit higher bids in an English auction than in a Dutch auction, so English auctions would generate more revenue when there are affiliated signals.
- This result is referred to as the "linkage principle".

Horse-house's management has been in the business for many years and has noticed that bidders at racehorse auctions tend to be quite superstitious. As soon as they have a good feeling about a horse, bidders generally, become quite risk-averse in their quest to win it.

Horse-house asks you, the stable boy/girl, whether and how you think this observation impacts the decision of whether to transition to English auctions.

3c. Given the potential risk aversion of bidders (and ignoring the common value element), would you expect English auctions to generate more, less or the same revenue as Dutch auctions? Why? (a qualitative explanation is sufficient)

- The potential risk aversion of bidders implies that Dutch auctions are likely to yield higher revenue than English auctions.
- Revenue equivalence tells us that English auctions and Dutch auctions yield the same revenue on average (for private values) when bidders are risk neutral.
- If bidders are risk averse, however, revenue equivalence breaks down.

- Risk aversion has no impact on the optimal bidding strategy in an English auction, which is still to remain in the auction until the price hits your value.
- Risk aversion does have an impact on the optimal bidding strategy in a Dutch auction. Whereas risk-neutral bidders would enter the auction when the price reaches $E(Y_1|Y_1 < x)$ (for single-shot games), risk averse bidders are likely to enter the auction at an earlier point, once the price has gone past their valuation, i.e. at some point between x and $E(Y_1|Y_1 < x)$. This is because a risk-averse bidder has a concave utility function which places relatively less value on the potential to generate high surplus from realising a lower price. It is therefore better to enter the auction earlier than a risk neutral bidder would have done, i.e. submit higher bids.
- We thus know that:
    - Dutch auctions and English auctions yield the same revenue for risk neutrality (the revenue equivalence theorem).
    - Introducing risk aversion does not affect bidder behaviour in English auctions but makes them submit higher bids in Dutch auctions.
- Thus, it must be that Dutch auctions generate higher revenue than English auctions when bidders are risk averse.

Regardless of your advice (and that of the accountant), Horse-house ultimately decides to stick with its tried-and-tested Dutch auction format (since the Dutch format is such an integral part of the Horse-house brand).

Horse-house is now arranging the sale of two young colts (i.e. young male horses): Thunder and Lightning. Thunder and Lightning are twins and are deemed to have exactly equal potential as racehorses. Horse-house is going to sell the two colts in two sequential Dutch auctions. In each auction, the price will start at 1 million pounds and then slowly descend until a bidder submits a bid, at which point the horse will be sold at that price.

Three bidders will take part at the auctions: Tommy, Charlie and Alfie. Tommy, Charlie and Alfie each have single-unit demand, i.e. each of them wants to buy only one of the two horses. Each bidder holds a valuation for one of the colts which is uniformly distributed between 0 and 1 million pounds. Each of the bidders is indifferent between the two horses. For the purposes of the rest of this problem, you can ignore the potential common value element and also assume bidder risk neutrality.

3d. Find the symmetric equilibrium bidding strategy in each of the two sequential auctions. Please also explain the intuition of your result.

- We know that the equilibrium bidding strategy for auction number k in a series of K sequential first-price auctions is to bid $\beta_k(x) = E(Y_K|Y_k < x < Y_{k-1})$.
- In other words, it is always optimal to bid the expected value of the bidder with the K-th highest value among other bidders, conditional on you yourself having the highest remaining value.
- The strategically equivalent bidding strategy in a Dutch auction is to bid in the following way:
$$\beta_k(x) = enter\ the\ bidding\ when\ the\ price\ reaches\ E(Y_K|Y_k < x < Y_{k-1})$$
- In the second auction, and given a uniform distribution between 0 and 1 million pounds, the symmetric equilibrium bidding strategy is thus to enter the bidding when the price reaches:
$$E(Y_2|Y_2 < x < Y_1) = \frac{(N-1)+1-2}{(N-1)+1}(x - \underline{x}) = \frac{(2-1)+1-2}{(2-1)+1}(x-0) = \frac{1}{2}x$$
, where we insert for $N = 2$ to reflect that the total number of bidders in round 2 (including yourself) is equal to 2.
- In the first auction, the symmetric equilibrium bidding strategy is to enter the bidding when the price reaches:

$$E(Y_2|Y_1 < x) = \frac{(N-1)+1-2}{(N-1)+1}(x - \underline{x}) = \frac{(3-1)+1-2}{(3-1)+1}(x-0) = \frac{1}{3}x$$

, where we insert for $N = 3$ to reflect that the total number of bidders in round 1 (including yourself) is equal to 3.

- Hence, we find the following symmetric equilibrium bidding strategy for the series of two auctions:

$$\beta_1(x) = enter\ the\ bidding\ when\ the\ price\ reaches\ \frac{1}{3}x$$
$$\beta_2(x) = enter\ the\ bidding\ when\ the\ price\ reaches\ \frac{1}{2}x$$

- Intuition:
  - Bidders will shade in the second auction in the same way as in a regular first-price auction (given the number of competitors that are left). This is because the second auction is the last auction in the series and is thus essentially equivalent to a one-shot game. Given two bidders and a uniform distribution, it is thus optimal to enter the bidding when the price reaches half your value. This point balances the probability of winning (which is greater when entering the bidding at a high price) with the surplus in the case of a win (which is greater when entering the bidding at a low price).
  - Bidders will shade more in the first auction than they would do in a regular first-price auction (given the number of competitors that are left). They thus wait longer than usual and only enter the bidding when the price reaches a third of their value. The reason why it is optimal to wait a little longer than usual is essentially because they know that they will have another chance to win in the following auction. The bidders thus have two reasons to shade in the first auction: 1) the standard first-price shade to generate surplus and 2) a shade because there will be more opportunities to win the item later and there is no point "over-paying" now.

Horse-house proceeds with the first auction.

To everyone's shock, Thunder is sold to Tommy in the first auction at a price of just 1 pound! Horse-house suspects that the three bidders may have formed a bidding ring, i.e. they may have agreed not to compete with one another at the live auction.

Horse-house wonders how it can counter such a bidding ring and considers implementing a reserve price for the auction of Lightning. Horse-house wants to set the revenue-maximising reserve price in the second auction, under the assumption that there is a bidding ring, and assuming that Tommy was the highest value bidder.

Horse-house's management asks you, the stable boy/girl, which reserve price it should set.

3e. What is the revenue-maximising reserve price in the second auction? Hint: you may need to use Wolfram Alpha (or similar) to identify the roots of a polynomial.

- Although this is a Dutch auction, we will calculate the revenue-maximising reserve price for a sealed-bid second-price auction. We can do this because there is revenue equivalence (given private values), which means that the same reserve price which maximises revenue for a sealed-bid second-price auction would also maximise revenue for a Dutch auction.
- Assuming that Tommy was the highest value bidder, the ring leader in the second auction will have a value equal to the highest remaining draw, which is the second highest draw, i.e. $Y_2^N = \max(X_2, X_3)$.
- Expected revenue in the second auction, in the presence of a bidding ring, as a function of the reserve price, is thus equivalent to:

$$E\big(R(r)\big) = prob(\max(X_2, X_3) > r) * r$$

, i.e. revenue is equivalent to the reserve price r, in the event that the bid submitted by the bidding ring (which is equivalent to the second-highest value) exceeds the reserve price r – and otherwise 0.

- To find the revenue-maximising reserve price, we need to maximise $E\big(R(r)\big)$ with respect to r. First, we will rewrite $prob(\max(X_2, X_3))$:
    - We know that the probability that $\max(X_2, X_3)$ exceeds r is equivalent to 1 minus the probability that $\max(X_2, X_3)$ is less than r, i.e.:
    - $prob(\max(X_2, X_3) > r) = 1 - prob(\max(X_2, X_3) < r)$
    - The probability that the second-highest draw out of three draws is less than r is equivalent to the probability that either:
        a) All three draws are less than r, which occurs with probability r^3 or
        b) Only the highest draw was above r, which occurs with probability $r^2 * (1 - r) * 3$ since one draw was above r, the other two were below, and this could happen in one of three ways (since any of the three bidders could be the one that drew above r)
    - We can thus rewrite $1 - prob(\max(X_2, X_3) < r)$ as $1 - (r^3 + r^2 * (1 - r) * 3)$.
- Inserting this into expected revenue, we get:

$$E\big(R(r)\big) = \big(1 - (r^3 + r^2 * (1 - r) * 3)\big) * r =$$
$$\big(1 - (r^3 + (r^2 - r^3) * 3)\big) * r = \big(1 - (r^3 + 3r^2 - 3r^3)\big) * r =$$
$$(1 + 2r^3 - 3r^2) * r = r + 2r^4 - 3r^3$$

- And taking a first-order condition to find the roots with respect to r, we get:

$$\frac{\partial E\big(R(r)\big)}{\partial r} = 1 + 8r^3 - 9r^2$$

- Wolfram Alpha tells us that the roots of this polynomial are at:
    - $r = 1$
    - $r = \frac{1}{16} + \frac{\sqrt{33}}{16} \cong 0.421$
    - $r = \frac{1}{16} - \frac{\sqrt{33}}{16} \cong -0.297$
- The only root which is on the range $[0:1]$ is 0.421, and thus this either minimises or maximises revenue on the value range. Visual inspection of the equation and the revenue function reveals that it must in fact be a maximum.
- The revenue-maximising reserve price is thus 0.421.
- We found in class that the revenue-maximising reserve price in the presence of a bidding ring with 2 bidders drawn from a uniform distribution on the range $[0:1]$ was $\frac{1}{\sqrt{3}} \cong 0.577$. It is intuitive that the reserve price would be lower in this case since the highest-value bidder has already won the first auction, and the distribution of the second-highest out of three draws will be lower than the highest out of two.
- *This was a difficult problem and we have given full or partial credit to decent and well-reasoned attempts at identifying the correct answer.*

Problem 4

Danish shipping company mAhoy is struggling financially. Shipping volumes have been declining due to the ongoing trade war between China and the US, and the company is faced with an oversupply of ships. To reduce supply and free up financial liquidity, the company has decided to divest one of its largest container carriers, Queen Margrethe II. At the same time, the company is planning an IPO in the near future to generate more capital and thereby weather the storm.

You are a student employed in the department tasked with selling the ship. Due to the upcoming IPO, mAhoy needs a good estimate of the expected selling price and the CFO has asked your department to come up with a number. For now, mAhoy's goal is to get as high a selling price as possible.

The team at your department has done some research about the prospective bidders. You expect N risk-neutral symmetric bidders to participate in the auction. Your department assumes that the bidders' values for the ship are interdependent and equal to:

$$v_i = \frac{1}{2}x_i + \frac{1}{2} * \frac{1}{N-1}\sum_{j \neq i} x_j$$

Where each bidder's signal, x_i, is uniformly distributed and independently between 0 and 100 million US dollars.

The head of your department is leaning towards selling the ship using a 2nd price auction. He has heard that with this format it is a dominant strategy for all of the bidders to bid truthfully – i.e. to place a bid of x_i. You are quite sure that he is incorrect, since you know of the winner's curse when bidders' values are interdependent.

To convince your boss, you have decided to do some calculations.

4a. Derive the bidders' expected payoff from winning the auction depending on the number of bidders in the auction, N, under the assumption that each bidder bids its signal x_i. How many bidders are needed for the expected payoff to become negative? Please also explain the intuition of your result.

- The bidders' expected payoff from winning the auction under the assumption that each bidder bids its signal is equal to the expected value given that the bidders wins the auction (i.e. has the highest signal) minus the expected price, which is equal to the second highest signal since all bidders are assumed to bid their signal in the second-price auction:

$$\begin{aligned} E[\pi_i | x_i = x, b_i = x, b_1 = x_1, \ldots, b_N = x_N] &= E[v | x_i = x, Y_1 < x] - E[Y_1 | Y_1 < x] \\ &= \left[\frac{1}{2}x + \frac{1}{2} * \frac{N-1}{N-1} * \frac{1}{2}x\right] - \left[\frac{N-1+1-1}{N-1+1}x\right] \\ &= \frac{3}{4}x - \frac{N-1}{N}x \end{aligned}$$

- The expected payoff is positive for $N < 4$, zero for $N = 4$ and negative for $N > 4$. With a negative expected payoff for $N > 4$ it cannot be a symmetric equilibrium bidding strategy for bidders to always just bid their own signal for any given N.
- In a second-price auction with interdependent values, it is optimal for bidders to shade their bids to account for the winner's curse. In a symmetric game with symmetric bidding strategies, bidders can infer that they have the highest signal of all bidders if they win the auction and will adjust the expected value of the item and their bid accordingly. In the assumed equilibrium above, bidders do not shade and will thus get a payoff that is decreasing in N.

You have managed to convince your boss and he is quite happy with your work. Thus, he has asked you to derive the correct symmetric equilibrium bidding strategy, i.e. what bidders should actually bid (given that they do not simply bid x_i).

4b. Derive the bidders' symmetric equilibrium strategy and the degree of the shading depending on N. Please also explain the intuition of your result.

- With interdependent values in a second-price auction (and the other standard assumptions, risk neutrality etc.) there exists a symmetric equilibrium strategy where all bidders bid the expected value $b = v(x, x)$:

$$\begin{aligned} v(x,x) &= \frac{1}{2}x + \frac{1}{2}\frac{1}{N-1}x + \frac{1}{2}\frac{N-2}{N-1}\frac{1}{2}x \\ &= \frac{2(N-1)}{4(N-1)}x + \frac{2}{4(N-1)}x + \frac{N-2}{4(N-1)}x \\ &= \frac{2N-2+2+N-2}{4(N-1)}x \\ &= x\frac{3N-2}{4N-4} \end{aligned}$$

- The above bids represent the optimal degree of shading to account for the winner's curse, which is increasing in N. The risk of the winner's curse is increasing in N, since it is "worse news" having the highest signal of e.g. 10 bidders than of 3 bidders. For $N = 2$, bidders do not shade because the expectation is equal to their own bids, but for $N > 2$ bidders shade.

Your department expects five bidders to participate in the auction.

4c. Derive expected revenue from selling the ship with five expected bidders.

- We know that with an increasing symmetric equilibrium strategy in a second-price auction, the bidder with the second highest signal will set the price. To derive the expected revenue, we thus start by finding the expected second highest signal between 0 and 100M:

$$\frac{N+1-2}{N+1}100M = \frac{N-1}{N+1}100M$$

- Which can be plugged into the symmetric equilibrium bidding strategy:

$$100M\frac{N-1}{N+1}\frac{3N-2}{4N-4} = 100M\frac{3N-2}{4N+4}$$

- Which for $N = 5$ is equal to:

$$100M\frac{15-2}{20+4} = 100M\frac{13}{24} \approx 54M$$

Your boss has heard that an open auction format such as the English auction is preferred when bidders' values are interdependent for two reasons in particular. Firstly, an open format will lead to increased expected revenue due to price discovery. Secondly, an open format reduces strategic complexity for the bidders. Impressed by your work so far, he has asked for your opinion.

4d. Are the two statements correct given the assumptions about the prospective bidders? Does this conclusion hinge on a specific assumption made by your department? (A qualitative explanation is sufficient)

- The first statement is incorrect due to an assumption of independent signals in this model. Revenue equivalence only breaks down with affiliated signals, whereas with interdependent values and independent signals, revenue equivalence still holds.
- The second statement is correct. The English format has a stronger equilibrium concept than the sealed-bid second-price format, which is the *ex post* equilibrium, compared to the Bayesian-Nash equilibrium (i.e. *ex ante)* for the sealed-bid second-price format. The ex-post equilibrium has a so-called "no regret" feature, which makes it less complex to bid, since bidders do not have to average their bid across many different scenarios, but simply compute whether the expected value of winning exceeds the current price at all prices. Thus, no bidders will regret their bids after the auction is concluded, as long as all bidders play the symmetric equilibrium bidding strategy. This means that in the equilibrium, it is not possible to lose money in an open format from incorrectly guessing the distribution of draws, which is not the case in the sealed-bid format.

The CFO has called you and your boss in for a meeting to present your results. She is impressed with your work, but wants to move forward with a 1st price sealed-bid auction since she is worried that only one or two bidders will participate in the auction due to the dire market circumstances facing all shipping companies.

At the meeting, the CFO has also presented you with a new complexity. The board of mAhoy is worried that some of the bidders will not take proper care of the ship. Queen Margrethe II is one of mAhoy's finest ships and she was the last ship to be produced at mAhoy's shipyard, which was put out of operation in the wake of the global financial crisis. Thus, the board believes that it is important for mAhoy's reputation that the ship is kept in fine shape by her new owner. The CFO wants you to think of a way to incorporate this goal into your design of the 1st price auction.

To incorporate this concern into the design, your department has devised a multi-criteria auction (you have experience using this approach from procurement auctions).

The design will work as follows: All prospective bidders will be assigned a score between 0 and 10 depending on how well mAhoy thinks the company will maintain the ship. 10 is the highest score. The price bids will also be translated into a linear score between 0 and 10, where 10 will be awarded to bids of 100m and 0 will be awarded to bids of 0. The two scores will then be combined to produce a total score, S_i^t:

$$S_i^t = \frac{1}{2} S_i^b(b_i) + \frac{1}{2} S_i^m$$

Where $S_i^b(b_i)$ denotes the price bid score for bidder i as a function of the price bid and S_i^m denotes the maintenance score for bidder i. The bidder with the highest total score will be awarded the ship and will pay its own price bid, b_i. The maintenance scores will be published prior to the auction.

4e. How would the multi-criteria auction outlined above affect price bidding behaviour for hypothetical bidders with relatively low maintenance scores or relatively high maintenance scores respectively? (a qualitative explanation is sufficient)

- The format has changed from second-price to first-price format. We know that in a standard first-price auction with interdependent values, bidders will shade for two reasons: 1) because they set their own price and 2) to account for the winner's curse.

- The introduction of the maintenance scores will impact the extent of the optimal shade in the second point above, since each bidder will now be more/less likely to win the auction with a given price bid, depending on whether they have a relatively high/low maintenance score. Relatively weak bidders will bid more aggressively, i.e. shade less, and relatively strong bidders will bid less aggressively, i.e. shade more. This can be compared to the standard asymmetry result with private values (although it is not exactly identical).
- The scores will not affect "winner's curse" shade as the advantage is only through probability of winning and not actual value/signal.

Problem 5

The municipality of Copenhagen has received many complaints from its citizens over a series of new companies that offer short-term rentals of electric scooters (e-scooters). The majority of the complaints relate to an issue of overcrowded sidewalks throughout the city. To mitigate this issue, the city council has decided to introduce a licensing system to restrict the number of e-scooters in the city to 1,000 units. The licenses will expire after five years, after which the municipality will assign new licenses.

There are currently five companies offering the rental service in Copenhagen. Since the e-scooter rental service is a new and developing industry, the municipality has limited knowledge of how profitable the companies are today and how profitable the companies will be depending on the number of licenses they are allocated. At this stage, the city council is also unsure of its main considerations/goals in relation to the auction other than that it wants a well-functioning low-priced rental service.

The municipality has decided to allocate the licenses to the rental companies using an auction and has asked you to submit a proposal for auction expert assistance. In the proposal, you are asked to write:

- A short of description of the main characteristics of the licenses and prospective bidders as well as what that means for auction design
- Your opinion on what you think the main goal(s) for the municipality should be when designing the auction

5a. Write the proposal as outlined in the two bullets above in a maximum 250 words. (A qualitative explanation is sufficient.)

- A good answer is well-structured and mentions at least some of the following elements (but is not restricted to the below):
    - The municipality has 1,000 identical licenses for sale. This is thus a multi-unit allocation problem (and the municipality could choose to package the licenses in different ways).
    - A multi-unit allocation problem probably calls for either a simultaneous multi-unit auction or sequential auctions.
    - Values are not private. The pay-off from a license depends on the business case of the rental service, which is uncertain and, to some degree, the same for all bidders (e.g. the future popularity of e-scooters).
    - Licenses are not purely common value for the bidders either. The bidders will to some degree differ in branding, technology, costs, etc., making the business case of the licenses differ between bidders.

- Licenses will probably feature complementarity. In all likelihood, no bidder can make use of just a single license and the market for e-scooter rentals may feature scale economies, such that the costs are decreasing and the desirability is increasing in the number of licenses allocated and thereby e-scooters on the streets.
- Given that the municipality wants a well-functioning low-priced service, the main objective for the municipality should be efficiency, such that the bidder(s) with the best combination of low costs and high quality are allocated the licenses.
- A second goal could be competition on the e-scooter market ensuring that the efficiency gains are passed on to the consumers. If one bidder wins all licenses, the resulting monopoly could result in inefficiently high prices to the detriment of the consumers (and society).

After reading your proposal, the municipality has decided to hire you. The municipality has done some more research on the market for e-scooter rentals and can share two findings with you. Firstly, the municipality expects that prospective bidders will be highly uncertain regarding the business case of the rental service since the market is still immature. Secondly, the municipality expects that each company will require a certain scale of e-scooters to offer a profitable service.

5b. Based on the municipality's two findings, which auction format would you recommend? Why? (A qualitative explanation is sufficient.)

- With complementarity, a combinatorial format which allows for package bids is necessary, so that bidders can condition their bids for later units on successfully acquiring earlier units. Among sealed-bid formats, either the combinatorial Vickrey or combinatorial discriminatory could work. The combinatorial Vickrey is more likely to achieve efficiency which is presumably the main goal.
- With uncertain business cases, it seems like bidders will have affiliated signals meaning that they would benefit from a multi-round auction. This would bring higher revenue, which is probably not one of the municipality's explicit goals, but also lead to more efficiency, if bidders would have accounted for the winner's curse in different ways in a sealed-bid context.
- One of two formats is likely optimal:
    - The multi-round clock auction which allows for package bidding and allows for bidders to account for the winner's curse. A drawback is the uniform pricing mechanism which can lead to inefficiency since bidders will shade their bids on later units.
    - The sealed-bid combinatorial Vickrey which allows for package bidding and guarantees efficiency in a private value setting. In a common value setting, the combinatorial Vickrey may not guarantee efficiency, however, if bidders account for the winner's curse differently. A combinatorial Vickrey is also arguably more difficult to bid in than a clock auction since bidders must submit many bids at the same time instead of just a single bid per round at the round price.
- The municipality may consider implementing a reserve price, not to maximise revenue, but to avoid embarrassingly low revenue and to deter strategic bidding.
- The municipality may also consider implementing an activity rule and/or an information rule if a multi-round auction is implemented – again, to deter strategic bidding.

One of the electric scooter rental companies, Scoolectric, is infamous for employing an aggressive market strategy of swarming each city with scooters trying to push competitors out of the local market. The council is worried that Scoolectric will outbid all competitors in the auction to secure a monopoly in Copenhagen and subsequently push up rental prices.

5c. Could you address this concern using auction design? Could this affect the municipality's expected revenue? Could there be any other issues associated with this approach? (A qualitative explanation is sufficient)

- Bid caps or set-asides for new entrants (bid floors) would rule out a monopoly on the e-scooter rental market.
- These instruments would lead to lower expected revenue for the municipality and could pose the risk that some bidders will drop out of the auction because they cannot achieve minimum viable scale (if for instance the cap is lower than the licenses they need to run a profitable service). With 5 expected bidders, a cap of 200 or less will lead to revenues defined by the reserve price.
- These instruments would also potentially lead to inefficient outcomes to the detriment of consumers. For instance, it might in fact be the most efficient outcome that all licenses are allocated to a single bidder.
- This potential inefficient outcome and expected loss of revenue has to be weighed against the gain for consumers through lower prices and/or higher quality of the service.